*International Journal of Theoretical Physics*, Vol. 17, No. 4 (1978), pp. 287–292

# Some Topological Configurations in Gauge Theories[1]

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A Higgs vacuum field $\phi$ is characterized by the set of conditions $D_\mu\phi = 0$, which lead to a generalized Meissner effect and partially determine the vector potential $A_\mu$ in terms of $\phi$. Applying this method to the Weinberg–Salam theory, we assert that there exist stringlike configurations in which a pair of magnetic poles are bound by a flux string of the $Z^0$ field, with an energy scale in the TeV range. We also point out that pure gauges in non-Abelian gauge theories are not well-defined due to topological singularities. In order to be meaningful, they must be enlarged to a class of almost pure gauges which include the various known topological configurations.

## 1. INTRODUCTION

The main point of this paper is to discuss certain classical configurations (Nambu, to be published)[3] which seem possible in the $SU(2) \times U(1)$ theory of Weinberg and Salam, and which therefore are of direct physical interest in contrast, for example, to the Nielsen–Olesen string and the 't Hooft–Polyakov monopole, which primarily serve as prototypes of topological configurations.

By way of introduction, however, I will start with the general methods that will be used. Let us denote by $\phi(x)$ the nonzero vacuum expectation value of a Higgs field. The component index is suppressed for simplicity. Usually $\phi(x)$ is taken to be a constant, but the most general characterization of $\phi(x)$ is

$$D_\mu\phi = 0 \quad \text{for all } \mu \tag{1.1}$$

where $D_\mu$ is a covariant derivative. Equation (1.1) minimizes the kinetic and potential energies separately since it implies $|\phi|^2 = \text{const}$, but it also leads to consistency conditions

$$[D_\mu, D_\nu]\phi \propto F_{\mu\nu}\phi = 0 \quad \text{for all } (\mu\nu) \tag{1.2}$$

[1] Work supported in part by the U.S. Department of Energy, San Francisco Office, and by the Sherman Fairchild Distinguished Scholars Program.

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[3] Other relevant references are found in this paper.

0020-7748/78/0400-0287$05.00/0 

where $F_{\mu\nu}$ is a matrix acting on $\phi$. This means that det $F_{\mu\nu} = 0$ for all $(\mu\nu)$. In the case of $U(1)$, $\phi$ is one-dimensional, and $F_{\mu\nu} = 0$. Thus equation (1.2) may be regarded as a statement of the Meissner effect. For non-Abelian gauge theories, the corresponding generalized Meissner effect is not always perfect, or in other words, not all components of $F^i_{\mu\nu}$ are necessarily zero. If a linear combination $F^{i}_{\mu\nu}\lambda^i$ of generators can annihilate $\phi(x)$, these components $F^i_{\mu\nu}$ are not zero, and thus survive the Meissner effect. This happens in the 't Hooft–Polyakov $SO(3)$ model where $\phi$ is an isovector. It also happens in the $SU(2) \times U(1)$ theory of Weinberg and Salam, and the surviving component is nothing but the electromagnetic field.

Equation (1.2) thus amounts to a restriction on the vector potentials $A_\mu{}^i$ for a given $\phi$. Indeed it is possible to solve equation (1.1) for $A_\mu{}^i$. In the case of $SO(3)$, the equation

$$\partial_\mu \boldsymbol{\phi} - g\mathbf{A} \times \boldsymbol{\phi} = 0 \tag{1.3}$$

has the general solution

$$\begin{aligned} g\mathbf{A}_\mu &= \boldsymbol{\phi} \times \partial_\mu \boldsymbol{\phi} + a_\mu \boldsymbol{\phi} \qquad (\phi^2 = 1) \\ g\mathbf{F}_{\mu\nu} &= \partial_\mu \boldsymbol{\phi} \times \partial_\nu \boldsymbol{\phi} + f_{\mu\nu}\boldsymbol{\phi} \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \end{aligned} \tag{1.4}$$

where $a_\mu$ is still arbitrary.

Although equation (1.1) characterizes a Higgs vacuum, there are cases where the equations cannot be satisfied everywhere. This happens when the unit vector $\boldsymbol{\phi}(x)$ develops a topological singularity. Then $\boldsymbol{\phi}(x)$ is forced to vanish at the singularity, creating a 't Hooft–Polyakov monopole with a finite energy above the normal Higgs vacuum. More specifically, such will be the case if

$$\boldsymbol{\phi}(x) = (\mathbf{x}/|x|)f(x) \tag{1.5}$$

where $f(0) = 0, f(\infty) = 1$.

## 2. STRINGLIKE CONFIGURATIONS IN THE WEINBERG–SALAM THEORY

We will now turn to the Weinberg–Salam theory. If the doublet Higgs field is denoted by $u$, the analogs of equations (1.1) and (1.2) are

$$D_\mu u = [\partial_\mu - (ig/2)\mathbf{A}_\mu \cdot \boldsymbol{\tau} - (ig'/2)A_\mu{}^0]u = 0 \qquad [g\mathbf{F}_{\mu\nu}\cdot\boldsymbol{\tau} + g'F^0_{\mu\nu}]u = 0 \tag{2.1}$$

Manipulating with the first equation, we obtain

$$g\mathbf{A}_\mu(u^\dagger u) + g'A_\mu{}^0(u^\dagger \boldsymbol{\tau} u) + i(u^\dagger \boldsymbol{\tau}\, \partial_\mu u) = 0 \tag{2.2}$$

which can be transformed into many different forms because of the various Fierz identities involving $u$ and $u^\dagger$. In particular, $u^\dagger \boldsymbol{\tau} u \equiv \boldsymbol{\phi}$ has the same properties as the $\boldsymbol{\phi}$ used before, i.e., it is an isovector with $\boldsymbol{\phi}^2 = (u^\dagger u)^2$, which may be normalized to 1. In any case, equation (2.2) is equivalent to

$$g\mathbf{A}_\mu = -\boldsymbol{\phi} \times \partial_\mu \boldsymbol{\phi} - \xi i \boldsymbol{\phi}(u^\dagger \partial_\mu u) \qquad g' A_\mu{}^0 = -(1 - \xi)(u^\dagger \partial_\mu u) \tag{2.3}$$

with an arbitrary parameter $\xi$.

We would like to simulate the monopole situation given by equation 1.5). For this, we can take

$$u = \begin{pmatrix} \cos \theta/2 \\ \sin \theta/2 \quad e^{i\phi} \end{pmatrix} \tag{2.4}$$

in polar coordinates. However, this is not quite admissible along the entire negative $z$ axis because the phase of the second component becomes ill defined there, although $\boldsymbol{\phi}$ does not have such a singularity. Thus we are forced to make the second component vanish along the negative $z$ axis by modifying $u$ as

$$u = \begin{pmatrix} \cos \theta/2 \\ f(\rho) \sin \theta/2 \quad e^{i\phi} \end{pmatrix} \qquad f(0) = 0 \qquad \rho^2 = x^2 + y^2 \tag{2.5}$$

In other words, we have to create a semi-infinite Nielsen–Olesen string in addition to the monopole at the origin. Such a system will have infinite energy, but a finite-energy system can be created by joining a pair of monopoles with a string. This is very much like the meson in the string model where the quarks take the place of monopoles.

Since it seems impossible to find exact solutions, the existence of such configurations is an assertion based on the asymptotic behavior that we can handle, and an appeal to plausibility arguments. So we will not try to reproduce here further mathematical details, but state the basic results only. Some more details will be found in Nambu (1977). The dumbbell configuration under consideration contains fluxes of $SU(2)$ and $U(1)$ gauge fields. Each of them consists of a part which spreads out from the poles, and a part which is confined within the string. The sum of the two parts is such that each pole is a genuine $SU(2)$ monopole with quantized magnetic charge, whereas the $U(1)$ flux is solenoidal with no source. The combination of the $SU(2)$ and $U(1)$ fluxes outside of the string is the real magnetic field generated by the monopoles of charge

$$|Q| = (4\pi/e)(1 - \xi) \tag{2.6}$$

On the other hand, the string has the lowest energy when $\xi = \cos^2 \theta_W$, i.e., when it is made up of a combination corresponding to the $Z^0$ field which remains massless. (These statements apply to the long string limit. In general, the division of fluxes inside and outside the string depends on the dynamics of motion.)

The mass $M$ of each pole and the tension $\tau$ (energy per unit length) of the string are estimated to be

$$\begin{aligned} M &\approx (4\pi/3e)(\sin\theta_W)^{5/2}(m_H/m_W)^{1/2} \times (250\ \mathrm{GeV}) \\ \tau &= 1/2\pi\alpha' \approx \pi \cos\theta_W (m_H/m_W) x (250\ \mathrm{GeV})^2 \end{aligned} \tag{2.7}$$

Here $\alpha'$ represents the asymptotic slope of Regge trajectories generated by the rotating dumbbell. With $\sin^2\theta_W \approx \frac{1}{3}$, and an ad hoc ansatz $m_H = m_W$, both $M$ and $1/(\alpha')^{1/2}$ turn out to be $\sim$1 TeV.

Since this energy scale is much larger than $m_w \sim 0.1$ TeV, which controls the thickness of the string and the poles, the object is far from an idealized one-dimensional system. For our simple description to be valid, therefore, the length of the string must be at least comparable to $1/m_W$, which translates into an angular momentum $L \geqslant 70$. States having smaller $L$ are presumably ill defined and highly unstable.

For large $L$, the system will be relatively stable against electromagnetic or weak decay, but it is liable to break up, very much like the hadrons in the string model. The high $L$ states on the leading trajectory are inhibited from breaking up by the angular momentum barrier. This is because for a linear trajectory, a breakup of mass $M \rightarrow M_1 + M_2$ requires $M \geqslant M_1 + M_2$ or $L^{1/2} \geqslant L_1^{1/2} + L_2^{1/2}$, which is compatible with angular momentum conservation only if $L_1$ or $L_2 = 0$, or if the relative angular momentum is nonzero. By the same token, however, the formation of such states is also suppressed. Thus it is not clear whether well-defined dumbell states can actually be produced with any reasonable cross section in, for example, $p + p$ or $e^+ + e^-$ reactions. One might instead observe a broad enhancement in the yield of hadrons and leptons due to the formation of virtual dumbbell states.

## 3. ALMOST PURE GAUGES

As we have learned from the foregoing examples, it is often very useful to specify a vector potential $A_\mu$, under restricted conditions, in terms of a prepotential. In particular, a pure gauge is completely characterized by a unitary matrix $u$:

$$A_\mu = iu^\dagger \partial_\mu u \qquad u^\dagger u = uu^\dagger = 1 \tag{3.1}$$

As in the case of equation (1.1), however, such characterization is not as simple as one might think because $u$ can develop topological singularities. Take, for example, the case of $SU(2)$. The matrix $u$ may be parametrized in terms of a unit 4-vector

$$u(x) = \tau^a \hat{\phi}^a \qquad \tau^a = (\boldsymbol{\tau}, i) \qquad \hat{\phi}^2 = 1 \qquad A_\mu{}^i(x) = 2\eta^{ibc}\hat{\phi}^b \partial_\mu \hat{\phi}^c \tag{3.2}$$

where $\eta^{ibc}$ is a tensor introduced by 't Hooft. (This remark is due to S. Dimopoulos.) Consider now the ansatz

$$\hat{\phi}^a(x) = \hat{x}^a = x^a/|x| \tag{3.3}$$

It immediately leads to a singularity at the origin, and a nonzero Pontrjagin index. It does not seem proper to call such $A_\mu$ a pure gauge because

$$\tfrac{1}{2}\operatorname{Tr}\int F_{\mu\nu}\tilde{F}_{\mu\nu}d^4x = \int Q_\mu dS_\mu = 8\pi^2 \qquad Q_\mu = -\tfrac{2}{3}u^\dagger\partial_\nu u\partial_\lambda u^\dagger\partial_\rho u\epsilon^{\mu\nu\lambda\rho} \tag{3.4}$$

and hence $F_{\mu\nu}$ cannot be identically zero. We are thus persuaded to enlarge pure gauges to a class of almost pure gauges defined by

$$U = f(x)u \tag{3.5}$$

so that in the above example, $f(x)$ may be allowed to vanish at the origin, at the same time making $F_{\mu\nu} \neq 0$ in its neighborhood. This modification leads to

$$A_\mu = f^2u^\dagger\partial_\mu u \qquad F_{\mu\nu} = f^2(1-f^2)\partial_\mu u^\dagger\partial_\nu u + \partial_\mu f^2u^\dagger\partial_\nu u - (\mu \leftrightarrow \nu) \tag{3.6}$$

Note the interesting involutive symmetry $f^2 \leftrightarrow 1 - f^2$. With the ansatz, equation (3.3), and $f^2 = \frac{1}{2}$, we obtain the meron configuration of Callan et al. (1977). With $f^2 = x^2/(x^2 + \lambda^2)$, we obtain the pseudoparticle (instanton) of Belavin et al. (1975). Another ansatz

$$\hat{\phi}^a = (\mathbf{x}/|x|, 0) \tag{3.7}$$

with $f^2 = \frac{1}{2}$ gives the Wu–Yang monopole. A suitable function $f(x)$ will turn it into the 't Hooft–Polyakov monopole.

If we are further willing to consider superpositions of potentials of the form (3.6), we can also express 't Hooft's multi-instanton solution (Jackiw et al., 1977):

$$A_\mu = \sum f_i^2u_i^\dagger\partial_\mu u_i \qquad f_i^2 = (\lambda_i^2/x_i^2)\Big/\Big[\sum(\lambda_i^2/x_i^2) + c\Big] \tag{3.8}$$

It thus appears that almost pure gauges cover important topological configurations. Moreover, they are essential in studying the properties of the ground state, i.e., the vacuum. In fact, pure gauges are not a well-defined concept because it is impossible to separate them from their neighborhood consisting of almost pure gauges.[4] No doubt this lies at the heart of the problems recently posed by Gribov (1977).

We hope that reasonings along this line of thought will help elucidate the properties of the vacuum in quantum chromodynamics, and shed light on the question of quark confinement. More details will be presented elsewhere.

[4] A recent remark by I. M. Singer (private communication) regarding the topological nonfactorizability of the space of gauge fields is probably addressed to the same point.

## ACKNOWLEDGMENTS

I thank the hospitality of the California Institute of Technology where part of the present work was carried out. Also very profitable discussions with S. Dimopoulos are duly acknowledged.

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